ON SIERPIŃSKI SETS, HUREWICZ SPACES AND HILGERS FUNCTIONS

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Abstract. The Hurewicz property is a classical generalization of σ-compactness and Sierpiński sets (whose existence follows from CH) are standard examples of non-σ-compact Hurewicz spaces. We show, solving a problem stated by Szewczak and Tsaban in [ST1], that for each Sierpiński set S of cardinality at least $\mathfrak{b}$ there is a Hurewicz space H with $S \times H$ not Hurewicz.

Some other questions in the literature concerning this topic are also answered.

1. Introduction

The Hurewicz property, introduced in [Hu2], is a classical generalization of σ-compactness which attracted much attention in point-set topology and set theory (see [Ts] for an excellent self-contained introduction to this topic).

Sierpiński sets (whose existence follows from CH) provided first examples of non-σ-compact sests with the Hurewicz property.

In this article we present solutions to some open problems from the literature of the subject. In particular:

- Solving Problem 7.5 formulated by Szewczak and Tsaban in [ST1] we prove (cf. Theorem 3.1) that for every Sierpiński set $S \subseteq 2^{\mathbb{N}}$ of cardinality at least $\mathfrak{b}$ there is a Hurewicz space $H \subseteq 2^{\mathbb{N}}$ such that $S \times H$ is not a Hurewicz space.
- We provide a solution of the second part of Problem 3 stated by Banakh and Zdomskyy in [BZ] by showing (cf. Theorem 4.3) that, assuming $V = L$, there is a set in $2^{\mathbb{N}}$ with the Borel-separation property but without the Analytic-separation property.

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- We give a negative answer (cf. Theorem 5.1) to Problem 6.13 formulated by Sakai and Scheepers in [SS] by proving that, assuming CH, for each Sierpiński set $S \subseteq 2^{\mathbb{N}}$ there is a Hurewicz C-space $H \subseteq I^{\mathbb{N}}$ such that $S \times H$ is not a C-space.
- Solving Problem 6.6 formulated by Sakai and Scheepers in [SS] we confirm their conjecture by showing (cf. Theorem 5.4) that, assuming $\mathfrak{b} = \mathfrak{c}$, there is a C-space $E \subseteq I^{\mathbb{N}}$ such that E^n is a Hurewicz space for $n = 1, 2, \ldots$ but $E \times \mathbb{N}^{\mathbb{N}}$ is not a C-space.

Some key ideas of our approach go back to the seminal paper by E. Michael [Mich]. These ideas were used in the literature in connection with the Baire category, involving Lusin sets and Menger spaces, cf. the references in [PP].

The Hilgers functions, vital in some of our reasonings, are recalled in subsection 2.6.

2. Terminology and some auxiliary results

2.1. **Notation.** We identify the Cantor set with the product $2^{\mathbb{N}}$. The constant zero sequence in $2^{\mathbb{N}}$ is denoted by $\bar{0}$ and $\bar{Q}$ is the subspace of $2^{\mathbb{N}}$ consisting of all eventually zero sequences in $2^{\mathbb{N}}$ - a homeomorphic copy of the space of rationals $\mathbb{Q}$. Then $\bar{P}$ denotes $2^{\mathbb{N}} \setminus \bar{Q}$ - a homeomorphic copy of the Baire space $\mathbb{N}^{\mathbb{N}}$.

For a subset A of a product $X \times Y$, and $x \in X, y \in Y$, $A_x = \{y : (x, y) \in A\}$ is the vertical section of A at x, and the horizontal section $\{x : (x, y) \in A\}$ of A at y is denoted by A^y.

The smallest cardinality of a subset of $2^{\mathbb{N}}$ which is nonmeasurable with respect to the standard probability product measure λ on $2^{\mathbb{N}}$ is denoted by $\mathrm{non}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ denotes the smallest cardinality of a base of the σ-ideal of all λ-null sets (shortly: null sets). The smallest cardinality of a covering on $2^{\mathbb{N}}$ by null sets is denoted by $\mathrm{cov}(\mathcal{N})$.

The smallest cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is unbounded (dominating, respectively) in the ordering $\leq^*$ of eventual domination is denoted by $\mathfrak{b}$ ($\mathfrak{d}$, respectively), cf. [Bla].

2.2. **Sierpiński sets.** Let us recall that $S \subseteq 2^{\mathbb{N}}$ is a *Sierpiński set*, if it is uncountable and has countable intersection with every null set in $2^{\mathbb{N}}$, cf. [Mi]. More generally, for an uncountable cardinal κ we call $S \subseteq 2^{\mathbb{N}}$ a *κ-Sierpiński set*, if $|S| \geq \kappa$ and $|S \cap N| < \kappa$ for every null set

N in $2^{\mathbb{N}}$, cf. [MTZ]. In particular, a κ-Sierpiński set exists under the assumption that $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \kappa$.

We shall use the following observations which presumably belong to a folklore.

Lemma 2.1. *Let S be a Sierpiński set in $2^{\mathbb{N}}$. For any Borel set $B \subseteq 2^{\mathbb{N}}$ of positive measure there exists a Sierpiński set $E \subseteq B \setminus S$.*

Proof. Let us recall that there is a Borel isomorphism φ of $2^{\mathbb{N}}$ onto B, preserving λ-null sets, cf. [Ke, Theorem 17.41]. It follows that if $B \cap S$ is countable, then $E = \varphi(S) \setminus S$ is a required Sierpiński set. If, on the other hand, $B \cap S$ is uncountable, then $S_1 = \varphi^{-1}(B \cap S)$ is a Sierpiński set and if T is any Sierpiński set in $2^{\mathbb{N}}$ disjoint from S_1, then $E = \varphi(S_1)$ satisfies our requirements.

To check that, indeed, there is a Sierpiński set T in $2^{\mathbb{N}}$ disjoint from S, we shall split the argument into two cases according to the validity or non-validity of CH.

Case (A): $2^{\aleph_0} = \aleph_1$.

Let N_α, $\alpha < \omega_1$ be λ-null G_δ-sets in $2^{\mathbb{N}}$ such that every λ-null set in $2^{\mathbb{N}}$ is a subset of some N_α. We inductively choose

$$e_\alpha \in 2^{\mathbb{N}} \setminus (S \cup \bigcup_{\beta<\alpha} N_\beta \cup \{e_\beta : \beta < \alpha\}),$$

and then $T = \{e_\alpha : \alpha < \omega_1\}$ is a required Sierpiński set.

Case (B): $2^{\aleph_0} > \aleph_1$.

Let us take $M \subseteq S$ with $|M| = \aleph_1$. We claim that there is $x \in 2^{\mathbb{N}}$ such that if we let $T = x + M$, then $T \cap S = \emptyset$ and T is a required Sierpiński set. Indeed, otherwise $S \cap (x + M) \neq \emptyset$ for each $x \in 2^{\mathbb{N}}$ or, equivalently, $M - S = \bigcup_{x \in M}(x - S) = 2^{\mathbb{N}}$. But this is impossible, since the intersection of $M - S$ with any null set in $2^{\mathbb{N}}$ has cardinality at most $\aleph_1$. $\square$

Lemma 2.2. *Let S be a Sierpiński set in $2^{\mathbb{N}}$. Then there exists a Sierpiński set $T \subseteq 2^{\mathbb{N}} \setminus S$ of full outer measure $\lambda^*(T) = 1$.*

Proof. Let $\mathscr{E}$ be a maximal collection of pairs (E, E^*), where $E \subseteq 2^{\mathbb{N}} \setminus S$ is a Sierpiński set (cf. Lemma 2.1), E^* is a Borel set with $E \subseteq E^*$ and $\lambda(E^*) = \lambda^*(E)$, and for any distinct (E_1, E_1^*), (E_2, E_2^*) in $\mathscr{E}$, $E_1^* \cap E_1^* = \emptyset$. The family $\mathscr{E}$ is countable, and by Lemma 2.1, if we let $T = \bigcup\{E : (E, E^*) \in \mathscr{E}\}$, then T is a required Sierpiński set. $\square$

2.3. **Hurewicz and Menger spaces.** Let us recall that a separable metrizable space X is *a Hurewicz space*, if for each sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers of X, there are finite subfamilies $\mathcal{F}_n \subseteq \mathscr{U}_n$ such that $X = \bigcup_n \bigcap_{m \geq n}(\bigcup \mathcal{F}_m)$. By a theorem of Hurewicz (cf. [Hu2]) this is equivalent to the statement that for every continuous function $f : X \to \mathbb{R}^{\mathbb{N}}$ the image of X is bounded (in the sense that the subset of $\mathbb{N}^{\mathbb{N}}$ consisting of sequences of the form $(\lceil |f(x)(n)| \rceil)_{n\in\mathbb{N}}$ is bounded in the ordering $\leq^*$ of eventual domination). Any σ-compact space is a Hurewicz space and there exist (in ZFC) Hurewicz, non-σ-compact spaces (cf. subsection 2.4). A useful application of the above characterization is the fact that a separable metrizable space X which is a union of less than $\mathfrak{b}$ Hurewicz subspaces X_α, is Hurewicz, since its image under any continuous function $f : X \to \mathbb{R}^{\mathbb{N}}$ is bounded as the union of less than $\mathfrak{b}$ bounded sets of the form $f(X_\alpha)$; in particular, if $|X| < \mathfrak{b}$, then X is a Hurewicz space.

We shall often apply the following characterization of Hurewicz spaces (cf. [JMSS, Theorem 5.7]): *a subspace X of a compact, metrizable space K is a Hurewicz space if and only if for any G_δ-set G in K containing X, there is an σ-compact set F in K such that $X \subseteq F \subseteq G$.*

The characterization yields readily that any $\mathfrak{b}$-Sierpiński set S in $2^{\mathbb{N}}$ is a non-σ-compact Hurewicz space (note, however, that the existence of $\mathfrak{b}$-Sierpiński sets cannot be proved in ZFC). Indeed, if G is a G_δ-set in $2^{\mathbb{N}}$ containing S and $F_1 \subseteq G$ is a σ-compact set of measure $\lambda(G)$, then since S is a $\mathfrak{b}$-Sierpiński set, we have that $|S \setminus F_1| < \mathfrak{b}$. Consequently, $S \setminus F_1$ is a Hurewicz space so it can be covered by a σ-compact set $F_2 \subseteq G$ and if we let $F = F_1 \cup F_2$, then F is a σ-compact set with $S \subseteq F \subseteq G$ which witnesses that S is a Hurewicz space.

Let us also recall that a separable metrizable space X is *a Menger space*, if for each sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers of X, there are finite subfamilies $\mathcal{F}_n \subseteq \mathscr{U}_n$ such that $X = \bigcup_n(\bigcup \mathcal{F}_n)$ (cf. [Me], [Hu1]). Clearly, if X is σ-compact, then it is a Menger space and every Hurewicz space is also a Menger space but there are (in ZFC) Menger spaces which are not Hurewicz, cf [Ts].

2.4. **$\mathfrak{b}$-scale set $\mathfrak{B}$ of Bartoszyński and Shelah.** In [BS] Bartoszyński and Shelah gave a ZFC example of a Hurewicz, non-σ-compact subspace $\mathfrak{B}$ of $2^{\mathbb{N}}$ with the following properties:

(BS1) $\mathfrak{B}$ contains $\bar{Q}$ and has cardinality $|\mathfrak{B}| = \mathfrak{b}$;

(BS2) for each σ-compact subset K of $\bar{P}$, we have $|\mathfrak{B} \cap K| < \mathfrak{b}$ and $\mathfrak{B} \setminus K$ is a Hurewicz space.

More precisely, $\mathfrak{B}$ is the union of $\bar{Q}$ with a copy A (under a homeomorphism between $\mathbb{N}^{\mathbb{N}}$ and $\bar{P}$) of a $\mathfrak{b}$-*scale*, i.e., any well-ordered by eventual domination and unbounded subset of $\mathbb{N}^{\mathbb{N}}$.

Following [MTZ] we shall call $\mathfrak{B}$ *a* $\mathfrak{b}$-*scale set* (clearly, $\mathfrak{B} \setminus K$ is also a $\mathfrak{b}$-scale set, for any σ-compact $K \subseteq \bar{P}$).

A simpler proof that every $\mathfrak{b}$-scale set is a Hurewicz space can be found in [PZ, Remark 4.2], where the argument is based on the following observation concerning the set $\mathfrak{B}$ ((BS3) below implies (BS2), cf. [PZ, Remark 4.2], and the reverse implication follows readily from the characterization in subsection 2.3):

(BS3) for each σ-compact subset K of $\bar{P}$, there exists a σ-compact subset L of $2^{\mathbb{N}}$ such that $\bar{Q} \subseteq L$, $L \cap K = \emptyset$, and $|\mathfrak{B} \setminus L| < \mathfrak{b}$.

Note that condition (BS2) implies that the subspace $A = \mathfrak{B} \setminus \bar{Q}$ is not Hurewicz (cf. the characterization of Hurewicz spaces from subsection 2.3).

If $\mathfrak{b} = \mathfrak{d}$, then we can require that the set A used in the construction of space $\mathfrak{B}$ is a copy of a scale, i.e., a well-ordered by eventual domination, cofinal and unbounded subset of $\mathbb{N}^{\mathbb{N}}$. In this case, the subspace A of $\mathfrak{B}$ is not a Menger space, cf. [Ts, Lemma 1.4].

2.5. **λ-spaces and λ'-sets.** Let us recall that a subspace X of a compact metrizable space K is *a λ-space* if every countable set $N \subseteq X$ is relatively G_δ in X and is *a λ'-set in K* if every countable set $N \subseteq K$ is relatively G_δ in $X \cup N$.

Clearly, if X is a λ'-set in K, then it is a λ-space but the opposite implication is false (cf. [Mi, Theorem 5.6]). Let us recall the following two observations.

Lemma 2.3. *If X is a Hurewicz λ-space contained in a compact metrizable space K, then X is a λ'-set in K.*

Proof. Let as assume that X is a Hurewicz λ-space contained in K and let N be an arbitrary countable subset of K. Let $N_1 = N \cap X$ and $N_2 = N \setminus X$.

Since X is a λ-space, there is a G_δ-set G_1 in K such that $N_1 = G_1 \cap X$.

Since X is a Hurewicz space and $K \setminus N_2$ is a G_δ-set in K containing X, there is a G_δ set G_2 in K containing N_2 and disjoint from X.

Let $G = G_1 \cup G_2$. Then G is a G_δ-set in K and $N = G \cap (X \cup N)$. $\square$

Lemma 2.4. *If S is a Sierpiński set in $2^{\mathbb{N}}$, then S is a λ-space and hence a λ'-set in $2^{\mathbb{N}}$.*

Proof. Let N be a countable subset of S, and let G be a G_δ-set in $2^{\mathbb{N}}$ covering N, with $\lambda(G) = 0$. Then $G \cap S$ is a countable G_δ-set in S containing N, which readily implies that N is a G_δ-set in S. $\square$

2.6. **Hilgers functions.** In this note we shall frequently use the following diagonal construction, going back to Hilgers [Hi].

Let S be a subset of a set X, $A \subseteq X \times Y$ be a subset of the product $X \times Y$ of sets X, Y such that the projection $\pi_X(A)$ contains S, and let $\mathcal{F} = \{F_\alpha : \alpha < \mathfrak{c}\}$ be a family of subsets of $X \times Y$ with $S \subseteq \pi_X(F_\alpha)$ for $\alpha < \mathfrak{c}$. Given a partition $\mathcal{P} = \{S_\alpha : \alpha < \mathfrak{c}\}$ of the set S into non-empty sets such that $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$, we define a function $f : S \to Y$ in the following way: given $x \in S_\alpha$, where $\alpha < \mathfrak{c}$, we pick $f(x) \in A_x \setminus (F_\alpha)_x$, whenever such choice is possible, and we choose $f(x) \in A_x$ arbitrarily, if $A_x \subseteq (F_\alpha)_x$.

We shall say, cf. [PP], that f is a *Hilgers function* associated with the set A, the family $\mathcal{F}$ and the partition $\mathcal{P}$. If the partition $\mathcal{P}$ consists of singletons, we simply say that f is associated with A and $\mathcal{F}$.

One can easily verify that $Gr(f)$, the graph of f, has the following property

(H1) for any $\alpha < \mathfrak{c}$, if $Gr(f) \subseteq F_\alpha$, then $A_x \subseteq (F_\alpha)_x$ for all $x \in S_\alpha$.

3. No Sierpiński set of cardinality at least $\mathfrak{b}$ is productively Hurewicz

The following result provides a positive answer (in a strong form) to Problem 7.5 in [ST1] (repeated as Problem 5.5 in [ST2]).

Theorem 3.1. *For every Sierpiński set $S \subseteq 2^{\mathbb{N}}$ of cardinality at least $\mathfrak{b}$ there is a Hurewicz λ'-set H in $2^{\mathbb{N}}$ such that $S \times H$ is not Hurewicz; if $\mathfrak{b} = \mathfrak{d}$, one can have $S \times H$ not Menger.*

Proof. Let T be a Sierpiński set in $2^{\mathbb{N}}$ disjoint from S, with $\lambda^*(T) = 1$, given by Lemma 2.2 (recall that λ^* is the outer measure on $2^{\mathbb{N}}$).

We shall follow closely the reasoning from [PZ, Example 4.1 and Remark 4.2] concerning the $\mathfrak{b}$-scale set $\mathfrak{B}$ described in Subsection 2.4.

We put $A = \mathfrak{B} \setminus \bar{Q}$. Let S' be a proper subset of S of cardinality $\mathfrak{b}$, and let $g : S \to A$ map $S \setminus S'$ to a point $a \in A$, and S' bijectively onto $A \setminus \{a\}$.

We will prove that the set

$$H = Gr(g) \cup (T \times \bar{Q}) \subseteq (S \cup T) \times \mathfrak{B} \tag{3.1}$$

has the required properties (identifying $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $2^{\mathbb{N}}$, we can treat H as a subset of $2^{\mathbb{N}}$).

Let $\pi_i : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}},\ i = 1, 2$, denote the projections onto the first and second axis, respectively.

First, we will verify that

$$H \text{ is a Hurewicz space.} \tag{3.2}$$

To that end, fix a G_δ-set G in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing H. We have to find a σ-compact set lying between H and G (cf. subsection 2.3).

Since $\bigcap\{G^q : q \in \bar{Q}\}$ is a G_δ-set containing T, and T being a Sierpiński set, is a Hurewicz space (cf. subsection 2.3), there is a σ-compact set F with $T \subseteq F \subseteq \bigcap\{G^q : q \in \bar{Q}\}$. Therefore, for a σ-compact set $F \times \bar{Q}$ we have

$$T \times \bar{Q} \subseteq F \times \bar{Q} \subseteq G. \tag{3.3}$$

Let

$$K = \pi_2((F \times 2^{\mathbb{N}}) \setminus G). \tag{3.4}$$

Then K is a σ-compact set in $\bar{P} = 2^{\mathbb{N}} \setminus \bar{Q}$, cf. (3.3). By property (BS3) of $\mathfrak{B}$ (cf. subsection 2.4) we can find a σ-compact subset L of $2^{\mathbb{N}}$ such that

$$\bar{Q} \subseteq L \subset (2^{\mathbb{N}} \setminus K) \text{ and } |\mathfrak{B} \setminus L| < \mathfrak{b}. \tag{3.5}$$

By (3.3) and (3.4) we have

$$T \times \bar{Q} \subseteq F \times L \subseteq G. \tag{3.6}$$

Since $\lambda^*(T) = 1$ and $2^{\mathbb{N}} \setminus F$ is disjoint from T, we have $\lambda(2^{\mathbb{N}} \setminus F) = 0$, and S being a Sierpiński set, $|(2^{\mathbb{N}} \setminus F) \cap S| \le \aleph_0$. By the definition of H (cf. (3.1)), the set $N = [(2^{\mathbb{N}} \setminus F) \times 2^{\mathbb{N}}] \cap H$ is countable. Consequently, by (3.6), $F_1 = (F \times L) \cup N$ is a σ-compact subset of G containing $H \cap (2^{\mathbb{N}} \times L)$.

Finally, letting $X = H \setminus (2^{\mathbb{N}} \times L)$, by (3.1) and (3.5), we have $X \subseteq S \times (\mathfrak{B} \setminus L)$. Since S is a Sierpiński set and $|\mathfrak{B} \setminus L| < \mathfrak{b}$ (cf. (3.5)), X

is the union of less than $\mathfrak{b}$ Hurewicz sets of the form $X^y \times \{y\}$, where $y \in \mathfrak{B} \setminus L$ and $X^y \subseteq S$ (cf. subsection 2.3). It follows that X, being a Hurewicz space, is contained in a σ-compact subset F_2 of G. Therefore, $F_1 \cup F_2$ is a σ-compact subset of G containing H, witnessing (3.2).

To prove that H is a λ-space, it is enough to observe that H is a set with countable vertical sections H_x, $x \in S \cup T$, over a Sierpiński set. Indeed, if N be a countable subset of H, then $\pi_1(N)$ is a countable subset of $S \cup T$, so because $S \cup T$ is a λ-space (cf. Lemma 2.4), there is a G_δ-set L in $2^{\mathbb{N}}$ with $L \cap (S \cup T) = \pi_1(N)$. Then $L \times 2^{\mathbb{N}}$ is a G_δ-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ and $(L \times 2^{\mathbb{N}}) \cap H$ is a countable G_δ-set in H containing N.

Since we have already established that H is a Hurewicz space, it is a λ'-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, by Lemma 2.3.

It remains to check that $S \times H$ is not a Hurewicz space. But this follows readily from the fact that the product $S \times H$ contains a closed copy $\{(s,(x,y)) \in S \times H : s = x\}$ of $Gr(g)$ and $\pi_2(Gr(g)) = g(S) = A$ is not Hurewicz (cf. the end of subsection 2.4). If $\mathfrak{b} = \mathfrak{d}$, then the set A is not Menger (cf. [Ts, Lemma 1.4]), implying that the product $S \times H$ is also not Menger. □

4. A strengthening of the Hurewicz property

T. Banakh and L. Zdomskyy [BZ] considered the following property, stronger than the Hurewicz property: let $\mathscr{E}$ be a class of sets in a Polish space Z containing all G_δ-sets; $X \subseteq Z$ has *the $\mathscr{E}$-separation property* if for every $E \in \mathscr{E}$ containing X there is a σ-compact set containing X and contained in E.

The following two results are related to this notion.

The first one strengthens Theorem 2.1 in [SWZ]. Let us recall that a subset A of $2^{\mathbb{N}}$ is *perfectly meager in the transitive sense* (PMT for short, cf. [SWZ]) if, for every perfect subset P of $2^{\mathbb{N}}$, there exists an F_σ-set F in $2^{\mathbb{N}}$ such that $A \subseteq F$ and $F \cap (P + t)$ is meager in $P + t$ for each $t \in 2^{\mathbb{N}}$. Using the natural identification of the product of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $2^{\mathbb{N}}$, which preserves the algebraic and topological structure, we can apply this notion also to subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

Theorem 4.1. *For every κ-Sierpiński set $S \subseteq 2^{\mathbb{N}}$ of cardinality $\mathfrak{c}$, there is $T \subseteq S$, $|T| = \mathfrak{c}$, and a function $f : T \to 2^{\mathbb{N}}$ such that, whenever B is a Borel set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing the graph $Gr(f)$ of f, there is a*

σ-compact set F with $|T \setminus F| < \kappa$ and $F \times 2^{\mathbb{N}} \subseteq B$. In particular, $Gr(f)$ is not PMT and if $\kappa = \mathfrak{b}$, $Gr(f)$ is a Hurewicz λ'-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which projects onto $2^{\mathbb{N}}$.

We start the proof with the following technical lemma.

Lemma 4.2. *Let K be a compact subset of $2^{\mathbb{N}}$ of positive λ-measure, and let T be a subset of K intersecting each Borel (hence also each analytic) set in K of positive λ-measure in $\mathfrak{c}$ many points.*

Then, for any Polish space Y, there exists a function $f : T \to Y$ such that, whenever B is a Borel set in $2^{\mathbb{N}} \times Y$ containing $Gr(f)$, there is a σ-compact set $F \subset K$ with $\lambda(F) = \lambda(K)$ and $F \times Y \subseteq B$.

Proof. We can split T into sets T_α, $\alpha < \mathfrak{c}$, intersecting each Borel set in K of positive λ-measure.

Let $f : T \to Y$ be a Hilgers function associated with the set $A = K \times Y$, the family $\mathcal{F} = \{B_\alpha : \alpha < \mathfrak{c}\}$ of all Borel sets in $2^{\mathbb{N}} \times Y$ such that for any $\alpha < \mathfrak{c}$, $\pi_{2^{\mathbb{N}}}(B_\alpha)$ contains T, and the partition $\mathcal{P} = \{T_\alpha : \alpha < \mathfrak{c}\}$, cf. Section 2.6. We shall check that T and f have the required properties.

Let B be a Borel set in $2^{\mathbb{N}} \times Y$ containing $Gr(f)$.

Then $T \subseteq \pi_{2^{\mathbb{N}}}(B)$, so there is $\alpha < \mathfrak{c}$ such that $B = B_\alpha$ and, consequently, $Y \subseteq (B_\alpha)_x$ for all $x \in T_\alpha$, i.e., $T_\alpha \times Y \subseteq B_\alpha$ (cf. (H1)).

Then $\pi_{2^{\mathbb{N}}}((K \times Y) \setminus B_\alpha)$ is an analytic subset of K disjoint from T_α, hence it is a λ-null set. Let N be a λ-null G_δ-set containing $\pi_{2^{\mathbb{N}}}((K \times Y) \setminus B_\alpha)$ and let $F = K \setminus N$. Then F is σ-compact, $\lambda(F) = \lambda(K)$, and $F \times Y \subseteq B_\alpha = B$. □

Proof of Theorem 4.1. Let $\mathcal{A}$ be a maximal pairwise disjoint collection of Borel sets of positive λ-measure in $2^{\mathbb{N}}$ intersecting S in a set of cardinality less than $\mathfrak{c}$. Then $\mathcal{A}$ is countable, so $|S \cap \bigcup \mathcal{A}| < \mathfrak{c}$ and $|S \setminus \bigcup \mathcal{A}| = \mathfrak{c}$.

Let $T^* = 2^{\mathbb{N}} \setminus \bigcup \mathcal{A}$. Then $S \setminus \bigcup \mathcal{A}$ is a κ-Sierpiński set contained in T^*, hence $\lambda(T^*) > 0$. Moreover, by the maximality of $\mathcal{A}$, each Borel set in T^* of positive λ-measure intersects S in $\mathfrak{c}$ many points. In particular if we fix a compact set $K \subseteq T^*$ with $\lambda(K) > 0$ and let $T = K \cap S$, then $T \subseteq K$ is a κ-Sierpiński set of cardinality $\mathfrak{c}$ intersecting each analytic set in K of positive λ-measure in $\mathfrak{c}$ many points.

Let $f : T \to 2^{\mathbb{N}}$ be a function given by Lemma 4.2 applied for K, T and $Y = 2^{\mathbb{N}}$. We shall check that T and f have the required properties.

So let B be a Borel set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing $Gr(f)$, and let $F \subseteq K$ be a σ-compact set as in Lemma 4.2. Since $\lambda(K \setminus F) = 0$ and $T \subseteq K$ is a κ-Sierpiński set, $|T \setminus F| < \kappa$, as required.

To prove that $Gr(f)$ is not PMT consider $P = \{\bar{0}\} \times 2^{\mathbb{N}}$ and let B be any F_σ (or even Borel) set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing $Gr(f)$. Then, by what has already been proved, there exists $a \in 2^{\mathbb{N}}$ such that $\{a\} \times 2^{\mathbb{N}} \subseteq B$ and $B \cap ((a, \bar{0}) + P) = \{a\} \times 2^{\mathbb{N}}$ is not meager in $(a, \bar{0}) + P = \{a\} \times 2^{\mathbb{N}}$.

For the rest of the proof let us additionally assume that $\kappa = \mathfrak{b}$. We keep the notation from the first part of the proof.

From $|T \setminus F| < \kappa = \mathfrak{b}$ we have $|Gr(f) \setminus (F \times 2^{\mathbb{N}})| < \mathfrak{b}$, so $|Gr(f) \setminus (F \times 2^{\mathbb{N}})|$ is a Hurewicz space. It follows that if B is additionally assumed to be a G_δ-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing $Gr(f)$, we can cover $Gr(f) \setminus (F \times 2^{\mathbb{N}})$ by a σ-compact set L contained in B. Then $(F \times 2^{\mathbb{N}}) \cup L$ is a σ-compact set containing $Gr(f)$ and contained in B witnessing that $Gr(f)$ is a Hurewicz space.

To prove that $Gr(f)$ is a λ'-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, it is now enough by Lemma 2.3, to show that it is a λ-space. For this, let A be an arbitrary countable subset of $Gr(f)$. Repeating the argument from the first part of the proof we pick a λ-null G_δ-set N containing $\pi(A)$ and letting $F = 2^{\mathbb{N}} \setminus N$ we conclude that F is σ-compact set disjoint from $\pi(A)$ with $|T \setminus F| < \mathfrak{b}$. Consequently, letting $F_0 = F \times 2^{\mathbb{N}}$ we have $F_0 \subseteq (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus A$ and $|(Gr(f) \setminus (F_0 \cup A)| < \mathfrak{b}$, so there is an F_σ-set F_1 in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $(Gr(f) \setminus (F_0 \cup A) \subseteq F_1 \subseteq (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus A$. Finally, let $F = F_0 \cup F_1$. Then $Gr(f) \setminus A = F \cap Gr(f)$ which shows that A is a G_δ-set in $Gr(f)$.

Clearly, $Gr(f)$ projects onto the second coordinate $2^{\mathbb{N}}$, since otherwise we would have $Gr(f) \subseteq 2^{\mathbb{N}} \times (2^{\mathbb{N}} \setminus \{a\})$ for some $a \in 2^{\mathbb{N}}$, and then $B = 2^{\mathbb{N}} \times (2^{\mathbb{N}} \setminus \{a\})$ would be a Borel set containing $Gr(f)$ and $B_x \neq 2^{\mathbb{N}}$ for all x, contrary to the earlier conclusion. □

The second result, where $\mathscr{B}$ ($\mathcal{A}$, respectively) stands for the family of Borel (analytic, respectively) sets in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, provides a solution (under $V = L$) of the second part of Problem 3 in [BZ].

Theorem 4.3. *Assuming $V = L$, there is a set in $2^{\mathbb{N}}$ with the $\mathscr{B}$-separation property but without the $\mathcal{A}$-separation property.*

In the proof we shall use the following fact essentially due to Gödel. Let $\pi : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ denote the projection onto the first axis.

Lemma 4.4. *Assuming $V = L$, there is a coanalytic set $C \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that*

(i) $\pi(C) = 2^{\mathbb{N}}$ and $|\pi^{-1}(t) \cap C| = 1$ for each $t \in 2^{\mathbb{N}}$,
(ii) for every analytic set $E \subseteq C$, $\pi(E)$ is countable.

Proof. Let $W_0 \subseteq 2^{\mathbb{N}}$ be a $\mathbf{\Delta}^1_2$ Bernstein set (cf. [Ke, Example 8.24]) which means that neither W_0 nor $W_1 = 2^{\mathbb{N}} \setminus W_0$ contains a non-empty perfect subset of $2^{\mathbb{N}}$ (the proof of the existence of such a set under $V = L$ is based on the work of Gödel, cf. [Kh]).

By the Novikov-Kondô uniformization theorem (cf. [Ke, Theorem 36.14]), there are disjoint coanalytic sets $C_i \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, $i = 0, 1$, with $\pi(C_i) = W_i$ and $|\pi^{-1}(t) \cap C_i| = 1$ for each $t \in W_i$. Then $C = C_0 \cup C_1$ has required properties.

Indeed, if $E \subseteq C$ is an analytic set, then $E_i = E \cap C_i = E \cap ((2^N \times 2^N) \setminus C_{i-1})$ is an analytic subset of C_i. Consequently, $\pi(E \cap C_i)$ is an analytic subset of W_i. Since W_i is a Bernstein set, $\pi(E \cap C_i)$ is countable, and so is $\pi(E \cap C) = \pi(E \cap C_1) \cup \pi(E \cap C_2)$. □

We now proceed to the proof of Theorem 4.3.

Proof. With the help of CH we first inductively construct a Sierpiński set S in $2^{\mathbb{N}}$ which intersects every analytic non-null set in $2^{\mathbb{N}}$, then we partition it into non-empty sets S_α, $\alpha < \omega_1$, with the same property, and finally we let $\mathcal{P} = \{S_\alpha : \alpha < \omega_1\}$.

Let $A = (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus C$ where C is the set given by Lemma 4.4, and let $\mathcal{F} = \{B_\alpha : \alpha < \omega_1\}$ be the family of all Borel sets in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that, for any $\alpha < \omega_1$, $\pi(B_\alpha)$ contains S.

Let $f : S \to 2^{\mathbb{N}}$ be a Hilgers function associated with the set A, the family $\mathcal{F}$ and the partition $\mathcal{P}$, cf. Section 2.6. We shall check that $Gr(f)$ has the required properties.

To show that $Gr(f)$ has the $\mathscr{B}$-separation property let us consider a Borel set B in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing $Gr(f)$. Then there is $\alpha < \omega_1$ such that $B = B_\alpha$ and, consequently, $A_x \subseteq (B_\alpha)_x$ for all $x \in S_\alpha$, cf. (H1) in subsection 2.6. Therefore, the projection $\pi(A \setminus B)$ is an analytic set (A being analytic as the complement of C) disjoint from S_α, hence it is a λ-null set.

Let N_1 be a λ-null G_δ-set containing $\pi(A \setminus B)$ and let $F_1 = 2^{\mathbb{N}} \setminus N_1$. The set F_1 is σ-compact and $\pi^{-1}(F_1) \cap A \subseteq B$, which implies that the set $E = \pi^{-1}(F_1) \setminus B$ is contained in $(2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus A = C$. Since, clearly, E is Borel, $\pi(E)$ is countable, by Lemma 4.4.

Let N_2 be a λ-null G_δ-set in $2^{\mathbb{N}}$ containing $\pi(E)$ and let $F_2 = F_1 \setminus N_2$. Then the set F_2 is σ-compact, $\pi^{-1}(F_2) \subseteq B$ and, since $\lambda(F_2) = 1$ and S is a Sierpiński set, $|S \setminus F_2| \leq \aleph_0$. Consequently, $|Gr(f) \setminus \pi^{-1}(F_2)| \leq \aleph_0$, hence $F = \pi^{-1}(F_2) \cup (Gr(f) \setminus \pi^{-1}(F_2))$ is a σ-compact set containing $Gr(f)$ and contained in B witnessing that $Gr(f)$ has the $\mathscr{B}$-separation property.

The set A containing $Gr(f)$ is analytic and $A_x \neq 2^{\mathbb{N}}$ for all $x \in 2^{\mathbb{N}}$. Since, as we have already established, for each Borel set B containing $Gr(f)$ there is x with $B_x = 2^{\mathbb{N}}$ (in fact, $B_x = 2^{\mathbb{N}}$ for every $x \in F_2$), there is no Borel set containing $Gr(f)$ and contained in A.

In particular, $Gr(f)$ fails the $\mathcal{A}$-separation property. □

5. Products of C-spaces

Recall that a separable metrizable space X is a *C-space* if for every sequence $\mathscr{G}_1, \mathscr{G}_2, \ldots$ of open covers of X there is a sequence $\mathscr{H}_1, \mathscr{H}_2, \ldots$ of families of pairwise disjoint open subsets of X such that $\mathscr{H}_i$ refines $\mathscr{G}_i$ for $i = 1, 2, \ldots$ (i.e., each member of $\mathscr{H}_i$ is contained in a member of $\mathscr{G}_i$) and the union $\bigcup_{i=1}^{\infty} \mathscr{H}_i$ is a cover of X, cf. [Eng2]. Note that, when we want to verify that a subspace X of a metrizable space Z is a C-space, we can require that the families $\mathscr{G}_i, \mathscr{H}_i$ in the above definition consists of sets open in Z. This follows from the well-known fact that, for each disjoint collection $\mathscr{U}$ of open subsets of a subspace X of a metrizable space Z, there is a disjoint collection $\mathscr{U}'$ of open subsets of Z such that $\mathscr{U} = \{U \cap X : U \in \mathscr{U}'\}$.

The main results of this section address some problems concerning the products of C-spaces formulated by M. Sakai and M. Scheepers [SS]. The first one (Theorem 5.1) provides a negative answer to Problem 6.13 in [SS], while the second one (Theorem 5.4) is a negative solution of Problem 6.6 in [SS].

Theorem 5.1. *Assuming CH, for each Sierpiński set $S \subseteq 2^{\mathbb{N}}$ there is a Hurewicz C-space $H \subseteq I^{\mathbb{N}}$ such that $S \times H$ is not a C-space. Moreover, H is a λ-space.*

We will use the following simple observation (recall that a subspace of a C-space need not be a C-space).

Proposition 5.2. *Each Hurewicz subspace X of a compact metrizable C-space K is a C-space.*

Proof. Let $\mathscr{G}_1, \mathscr{G}_2, \dots$ be a sequence of open in K covers of X. Let $G = \bigcap\{\bigcup \mathscr{G}_i : i = 1, 2, \dots\}$. Since X is a Hurewicz space, we can find a σ-compact set F with $X \subseteq F \subseteq G$ (cf. subsection 2.3). The subspace F is a C-space, cf. [Eng2, proof of 6.3.8], therefore we can find a sequence $\mathscr{H}_1, \mathscr{H}_2, \dots$ of families of pairwise disjoint open subsets of F such that $\mathscr{H}_i$ refines $\mathscr{G}_i$ for $i = 1, 2, \dots$ and the union $\bigcup_{i=1}^{\infty} \mathscr{H}_i$ covers F, hence also X. □

We also need the next lemma.

Lemma 5.3. *Let X, Y be separable metrizable spaces, and let Z be a subspace of $X \times Y$ such that, for each open set V in $X \times Y$ containing Z, there is $x \in X$ with $\{x\} \times Y \subseteq V$.*

If Z is a C-space, then so is Y.

Proof. Let $\mathscr{W}_1, \mathscr{W}_2, \dots$ be a sequence of open covers of Y.

We define $\mathscr{U}_i = \{X \times W : W \in \mathscr{W}_i\}, i = 1, 2, \dots$. Using the C-space property of Z we can find open in $X \times Y$ disjoint collections $\mathscr{V}_1, \mathscr{V}_2, \dots$ such that $\mathscr{V}_i$ refines $\mathscr{U}_i$ and $Z \subseteq V = \bigcup\{\bigcup \mathscr{V}_i : i = 1, 2, \dots\}$. By our assumption on Z there is $x \in X$ with $\{x\} \times Y \subseteq V$. Then $\mathscr{D}_i = \{V_x : V \in \mathscr{V}_i\}$ is an open disjoint refinement of $\mathscr{W}_i$ and $Y \subseteq \bigcup\{\bigcup \mathscr{D}_i : i = 1, 2, \dots\}$. □

Proof of Theorem 5.1. There exists a metrizable compact C-space L containing a G_δ-subspace G which is not a C-space (cf. [Eng2, Example 6.1.21, Theorem 6.3.10 and Problem 6.3.D(b)]).

Let $\pi : 2^{\mathbb{N}} \times L \to 2^{\mathbb{N}}$ be the projection.

Let $T \subseteq 2^{\mathbb{N}}$ be a Sierpiński set disjoint from S, with $\lambda^*(T) = 1$, given by Lemma 2.2. Observe that, by CH, T intersects each Borel set in $2^{\mathbb{N}}$ of positive λ-measure in $\mathfrak{c}$ many points. Therefore, we can apply Lemma 4.2 for T, $K = 2^{\mathbb{N}}$ and $Y = L$, obtaining a function $f : T \to L$ such that

(5.1) for each Borel set $B \subseteq 2^{\mathbb{N}} \times L$ containing $Gr(f)$ there is a σ-compact set $F \subseteq 2^{\mathbb{N}}$ with $\lambda(F) = 1$ and $F \times L \subseteq B$.

Let $g : S \to G$ be a Hilgers function associated with the set $S \times G$ and the family $\mathcal{F} = \{V_\alpha : \alpha < \mathfrak{c}\}$ of all open sets in $2^{\mathbb{N}} \times L$ such that for any

$\alpha < \mathfrak{c}$, $\pi(V_\alpha)$ contains S. By condition (H1), cf. subsection 2.6, $Gr(g)$ has the the following property

$$\text{(5.2)} \qquad \text{for each open } V \subseteq 2^{\mathbb{N}} \times L \text{ containing } Gr(g) \text{ there is } s \in S \text{ such that } \{s\} \times G \subseteq V.$$

Finally, let

$$H = Gr(f) \cup Gr(g) \subseteq 2^{\mathbb{N}} \times L. \tag{5.3}$$

We claim that H has the desired properties.

First, we shall verify that H is a Hurewicz space. So, take a G_δ-subset B of $2^{\mathbb{N}} \times L$ containing H. By (5.1) we have a σ-compact set $F \subseteq 2^{\mathbb{N}}$ such that $\lambda(F) = 1$ and $F \times L \subseteq B$. The set $E = (S \cup T) \setminus F$ is countable, $S \cup T$ being a Sierpiński set. Therefore, the set $(F \times L) \cup [(E \times L) \cap H]$ is σ-compact, and we have

$$H \subseteq (F \times L) \cup [(E \times L) \cap H] \subseteq B. \tag{5.4}$$

Since the product $2^{\mathbb{N}} \times L$ is a C-space (cf. [Eng2, Theorem 6.3.11]), Proposition 5.2 assures us that so is H.

Next, note that condition (5.2) together with Lemma 5.3 implies that

$$\text{the graph } Gr(g) \text{ is not a } C\text{-space.} \tag{5.5}$$

Finally, observe that the product $S \times H$ contains a closed copy $\{(s,(x,y)) \in S \times H : s = x\}$ of $Gr(g)$, hence, by (5.5) $S \times H$ is not a C-space.

Moreover, since the projection onto the first axis maps H injectively into the λ-set $S \cup T$ ($S \cup T$ is a Sierpiński set, cf. Lemma 2.4), H is a λ-set. ☐

Theorem 5.4. *Assuming $\mathfrak{b} = \mathfrak{c}$, there is $E \subseteq I^{\mathbb{N}}$ such that*

(i) E is a C-space but $E \times \mathbb{N}^{\mathbb{N}}$ fails this property,

(ii) for each Hurewicz space $Y \subseteq I^{\mathbb{N}}$, $E \times Y$ is Hurewicz; in particular, E^n is Hurewicz for $n = 1, 2, \ldots$.

For the proof we will need two auxiliary facts. The first one is a simple observation extracted from the proof of properties of Example 2 in [EPo]. Recall that, for a cardinal number κ, a space X is *κ-concentrated about a set* $A \subseteq X$, if, for each open set in X containing A, we have $|X \setminus U| < \kappa$.

Proposition 5.5. *Let X be a separable metrizable space $\mathfrak{c}$-concentrated about a subspace $Y \subseteq X$. If Y is a C-space, then so is X.*

Proof. Let $\mathscr{W}_1, \mathscr{W}_2, \dots$ be a sequence of open covers of X.

By the C-space property of Y, we can find open in X disjoint collections $\mathscr{V}_2, \mathscr{V}_3, \dots$ such that $\mathscr{V}_i$ refines $\mathscr{W}_i$, $i = 2, 3, \dots$ and $Y \subseteq V = \bigcup\{\bigcup \mathscr{V}_i : i = 2, 3, \dots\}$. Since $|X \setminus V| < \mathfrak{c}$, the subspace $X \setminus V$ is zero-dimensional, hence there is an open in X disjoint collection $\mathscr{V}_1$ refining $\mathscr{W}_1$ and covering $X \setminus V$. Clearly, the sequence $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \dots$ witnesses the C-space property of X. □

In the next statement (probably a part of the folklore in this area), $\mathfrak{B}$ is a $\mathfrak{b}$-scale set described in subsection 2.4. This statement implies in particular, that for any Hurewicz space $Y \subseteq I^{\mathbb{N}}$, the product $\mathfrak{B} \times Y$ is Hurewicz, which was established by Miller, Tsaban and Zdomskyy in [MTZ, Theorem 6.7].

Lemma 5.6. *Let K be a metrizable compact space, $f : K \to 2^{\mathbb{N}}$ be a continuous map, and $E = A \cup C$ be a subspace of $f^{-1}(\mathfrak{B})$ such that $C = f^{-1}(\bar{Q}), A \cap C = \emptyset$, and $|A \cap f^{-1}(t)| = 1$ for all $t \in \mathfrak{B} \setminus \bar{Q}$.*

Then, for every Hurewicz space $Y \subseteq I^{\mathbb{N}}$, the product $E \times Y$ is Hurewicz.

Proof. Let $G \subseteq K \times I^{\mathbb{N}}$ be a G_δ-set containing $E \times Y$.

Let $\pi_1 : K \times I^{\mathbb{N}} \to K, \pi_2 : K \times I^{\mathbb{N}} \to I^{\mathbb{N}}$ be the projections.

The set $L' = \pi_2((C \times I^{\mathbb{N}}) \setminus G)$ is σ-compact and disjoint from Y. Since Y is Hurewicz, there is a σ-compact set $L \subseteq I^{\mathbb{N}}$ such that

$$Y \subseteq L \subseteq I^{\mathbb{N}} \setminus L'. \tag{5.6}$$

Then

$$C \times Y \subseteq C \times L \subseteq G. \tag{5.7}$$

Therefore, the set

$$M = \pi_1((K \times L) \setminus G) \text{ is } \sigma\text{-compact, and } M \cap C = \emptyset\,. \tag{5.8}$$

Since $f(M)$ is a σ-compact subset of $\bar{P}$, by property (BS3) of $\mathfrak{B}$ (cf. subsection 2.4), there is a σ-compact set N in $2^{\mathbb{N}}$ with

$$\bar{Q} \subseteq N \subseteq 2^{\mathbb{N}} \setminus f(M) \text{ and } |\mathfrak{B} \setminus N| < \mathfrak{b}. \tag{5.9}$$

Letting $F_1 = f^{-1}(N) \times L$, by (5.6), (5.8) and (5.9) we have

$$C \times Y \subseteq f^{-1}(N) \times L \subseteq F_1 \subseteq G \text{ and } |E \setminus f^{-1}(N)| < \mathfrak{b}. \tag{5.10}$$

Therefore, the product $(E \setminus f^{-1}(N)) \times Y$ is Hurewicz, as the union of less than $\mathfrak{b}$ Hurewicz sets of the form $\{x\} \times Y$, where $x \in E \setminus f^{-1}(N)$ (cf. subsection 2.3). Hence, we can find a σ-compact set F_2 with

$$(E \setminus f^{-1}(N)) \times Y \subseteq F_2 \subseteq G. \tag{5.11}$$

Finally, $F_1 \cup F_2$ is a σ-compact set containing $E \times Y$ and contained in G. □

Proof of Theorem 5.4. First, we will recall a (slightly modified) construction from [EPo, Example 2].

Let $L = 2^{\mathbb{N}} \times I^{\mathbb{N}}$, and let $\pi_0 : L \to 2^{\mathbb{N}},\ \pi_n : L \to I^n, n = 1, 2, \ldots$, be the projections defined by

$$\pi_0((x, (t_i)_{i=1}^{\infty})) = x\,; \tag{5.12}$$

$$\pi_n((x, (t_i)_{i=1}^{\infty})) = (t_1, t_2, \ldots, t_n)\,. \tag{5.13}$$

for $x \in 2^{\mathbb{N}}, (t_i)_{i=1}^{\infty} \in I^{\mathbb{N}}, n = 1, 2, \ldots$.

Fix a bijective enumeration $q_1, q_2, \ldots$ of the set $\bar{Q}$. We consider the closed equivalence relation on the space L which identifies points $(x, (t_i)_{i=1}^{\infty})$ and $(x', (t'_i)_{i=1}^{\infty})$, whenever there is $n \in \{1, 2, \ldots\}$ such that $x = x' = q_n$ and $\pi_n((q_n, (t_i)_{i=1}^{\infty})) = \pi_n((q_n, (t'_i)_{i=1}^{\infty})$. Denote the quotient compact space obtained in this way by K, and let $q : L \to K$ be the natural quotient map. Let $f : K \to 2^{\mathbb{N}}$ be the (unique) continuous map such that $f \circ q = \pi_0$.

Observe that

(5.14) $q|(\bar{P} \times I^{\mathbb{N}})$ is a homeomorphic embedding onto $f^{-1}(\bar{P})$;

(5.15) $f^{-1}(q_n)$ is homeomorphic with I^n, for $n = 1, 2, \ldots$

Let $g : \mathfrak{B} \setminus \bar{Q} \to I^{\mathbb{N}}$ be a Hilgers function associated with the set $(\mathfrak{B} \setminus \bar{Q}) \times I^{\mathbb{N}}$ and the family $\mathcal{F} = \{V_\alpha : \alpha < \mathfrak{c}\}$ of all open sets in $\bar{P} \times I^{\mathbb{N}}$ such that for any $\alpha < \mathfrak{c}$, $\pi_0(V_\alpha)$ contains $\mathfrak{B} \setminus \bar{Q}$. By condition (H1), cf. subsection 2.6, $Gr(g)$ has the the following property

(5.16) for each open $V \subseteq \bar{P} \times I^{\mathbb{N}}$ containing $Gr(g)$ there is
$x \in \mathfrak{B} \setminus \bar{Q}$ such that $\{x\} \times I^{\mathbb{N}} \subseteq V$.

Now, we can define our space E in the following way

$$E = A \cup C \subseteq K, \quad \text{where } A = q(Gr(g)) \text{ and } C = f^{-1}(\bar{Q})\,. \tag{5.17}$$

Clearly,

$$|A \cap f^{-1}(x)| = 1 \quad \text{for all } x \in \mathfrak{B} \setminus \bar{Q}\,. \tag{5.18}$$

Observe that the space E is $\mathfrak{b}$-concentrated about the set C. Indeed, if U is an open set in K containing C, then $f(K\setminus U)$ is a compact subset of $2^{\mathbb{N}}$ disjoint from $\bar{Q}$, hence, by property (BS2) of $\mathfrak{B}$ (cf. subsection 2.4), $|f(K\setminus U)\cap\mathfrak{B}| < \mathfrak{b}$. Consequently, $|E\setminus U| < \mathfrak{b}$, since f is one-to-one on A, cf. (5.18).

The set C is a C-space, being a countable union of finite-dimensional compacta, cf. (5.15), (5.17) and [Eng2, Theorem 6.3.8]. From Proposition 5.5 we conclude that E is a C-space.

Since the Hilbert cube $I^{\mathbb{N}}$ is not a C-space, Lemma 5.3 and condition (5.16) imply that the graph $Gr(g)$ is not a C-space, and neither is A, cf. (5.14). The space A homeomorphically embeds in $E \times \bar{P}$ as the closed subspace $\{(x, f(x)) :\ x \in A\} = \{(x, y) \in E \times \bar{P} :\ y = f(x)\}$, and since $\bar{P}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, we get (i) of the theorem.

Statement (ii) of the theorem follows immediately from Lemma 5.6, cf. conditions (5.17), (5.18). □

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